ON THE STABILITY OF ALMOST – PERIODIC MOTIONS

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A theorem due to Kamenkov [1] states that there exists an equivalence between the stability of periodic motions in the critical cases with nonessential singularity, and the stability of a steady-state motion in the critical case of a multiple zero root.

We extend this result to almost-periodic motions. We assume that the linear system is reducible and, that the almost-periodic coefficients of the right-hand sides of the differential equations of perturbed motion can be represented in terms of finite Fourier series with arbitrary frequency spectra.

The author of [2] shows the possibility of reducing the problem of stability of almost-periodic motions to the investigation of the stability of a steady-state motion, but he only deals with nonresonant cases. The present paper generalizes these results.

1. Let us consider a system of differential equations of perturbed motion, of the type

$$x_i^{\cdot} = \sum_{j=1}^{m} p_{ij}x_j + X_i(x_1, \ldots, x_m; t), \quad X_i(x_1, \ldots, x_m; t) = \sum_{l\geqslant 2}^{\infty} X_i^{(l)}(x_1, \ldots, x_m; t)$$
$$(i = 1, \ldots, m) \tag{1.1}$$

where p_{ij} are constant coefficients and X_i are holomorphic functions $x_1, \ldots, x_m$ with almost-periodic coefficients, which become zero when $x_1 = \ldots = x_m = 0$. We shall assume that the almost-periodic coefficients can be written as finite Fourier series with arbitrary frequency spectra.

As we know, any system whose nonlinear terms are of the form X_i and whose linear term coefficients are periodic functions of the same period, can be reduced to (1.1). This is also true in a number of cases when the linear term coefficients are almost-periodic.

We shall assume that the characteristic equation of the system (1.1) has k roots with negative real parts and n roots with real parts equal to zero. Of the latter, p roots are equal to zero and q pairs of roots are purely imaginary. In [2] it was shown that in the case with nonessential singularity, the system (1.1) can be replaced with an abbreviated system

$$y_s^{\cdot} = \sum_{k=1}^{n} g_{sk}y_k + Y_s(y_1, \ldots, y_n; t)$$
$$Y_s(y_1, \ldots, y_n; t) = \sum_{l\geqslant 2}^{N} Y_s^{(l)}(y_1, \ldots, y_n; t) \qquad (s = 1, \ldots, n) \tag{1.2}$$

in which g_{sk} are constants and Y_s have the same structure as X_i.

Eq. $|g_{sk} - \delta_{sk}\varkappa| = 0$ has p zero roots and q pairs of purely imaginary roots. The latter have no restrictions imposed on them, they can be simple or multiple and any number of the sets of solutions may correspond to them. We shall show, how, by means of a substitution given in [1], we can transform a subsystem of (1.2) with purely imaginary roots into another subsystem with a multiple zero root.

We shall first assume that the characteristic Eq. $|g_{sk} - \delta_{sk}\varkappa| = 0$ has only one pair of purely imaginary roots $\pm i\lambda$ of multiplicity r, with a single set of solutions corresponding to it. Linear substitution with constant coefficients transforms (1.2) into

$$z_s^{\cdot} = \sum_{k=1}^{n-2r} a_{sk}z_k + Z_s(z_i, \xi_\nu, \eta_\nu, t), \qquad \begin{aligned} \xi_j^{\cdot} &= -\lambda\eta_j + \sigma_{j-1}\xi_{j-1} + \Xi_j(z_i, \xi_\nu, \eta_\nu, t) \\ \eta_j^{\cdot} &= \lambda\xi_j + \sigma_{j-1}\eta_{j-1} + H_j(z_i, \xi_\nu, \eta_\nu, t) \end{aligned} \tag{1.3}$$

$$(s, i = 1, \ldots, n-2r; \quad j, \nu = 1, \ldots, r; \quad \sigma_0 = 0)$$

Eq. $|a_{sk} - \delta_{sk}\varkappa| = 0$ has a $(n-2r)$-tuple zero root. All σ_{j-1} can be assumed arbitrary since the values $\sigma_1, \ldots, \sigma_{r-1}$ can be varied using the substitution

$$\xi_2 = \alpha_2 x_2, \ \eta_2 = \alpha_2 y_2, \ldots, \ \xi_r = \alpha_r x_r, \ \eta_r = \alpha_r y_r$$

followed by a suitable choice of $\alpha_2, \ldots, \alpha_r$. Retaining the previous notation for the variables, we shall assume that all σ_{j-1} are equal to λ.

Introducing the substitution

$$\xi_1 = x_1\cos\lambda t + y_1\sin\lambda t, \ \xi_j = x_j\cos\lambda t + y_j\sin\lambda t + x_{j-1}\cos\lambda t + y_{j-1}\sin\lambda t$$

$$\eta_1 = x_1\sin\lambda t - y_1\cos\lambda t, \qquad \eta_j = x_j\sin\lambda t - y_j\cos\lambda t + x_{j-1}\sin\lambda t - y_{j-1}\cos\lambda t \tag{1.4}$$

$$(j = 2, \ldots, r)$$

we obtain

$$z_s^{\cdot} = \sum_{k=1}^{n-2r} a_{sk}z_k + Z_{s1}(z_i, x_\nu, y_\nu, t) \tag{1.5}$$

$$x_1^{\cdot} = X_1(z_i, x_\nu, y_\nu, t), \qquad x_j^{\cdot} = \lambda x_{j-1} + X_j(z_i, x_\nu, y_\nu, t)$$

$$y_1^{\cdot} = Y_1(z_i, x_\nu, y_\nu, t), \qquad y_j^{\cdot} = \lambda y_{j-1} + Y_j(z_i, x_\nu, y_\nu, t)$$

$$(s, i = 1, \ldots, n-2r; \quad j = 2, \ldots, r; \quad \nu = 1, \ldots, r)$$

We easily see that in the above system the functions Z_{s1}, X_j and Y_j all have the structure of X_i.

Characteristic equation of the subsystem in the variables x_j and y_j has a $2r$-tuple zero root with the corresponding two sets of solutions. If k sets of solutions ($k \leqslant r$) corresponded to the multiple root $\pm i\lambda$, then the transformation (1.4) would follow the same course for each set and, in the transformed system, the additional $2r$-tuple zero root would have the corresponding $2k$ sets of solutions.

In the general case of several pairs of purely imaginary roots, analogous manipulations can be performed for each pair of simple roots, or for multiple, purely imaginary roots. This will result in a system of the same n-th order, and the number of the corresponding sets of solutions will be determined by the character of the purely imaginary roots and by the number of sets of solutions for the multiple zero root of (1.2). We can write this system as

$$y_s^{\cdot} = \varkappa_{s-1}y_{s-1} + Y_s(y_1, \ldots, y_n; t) \qquad (s = 1, \ldots, n; \ \varkappa_0 = 0)$$

$$Y_s = \sum_{l \geqslant 2}^{N} Y_s^{(l)}(y_1, \ldots, y_n; t), \qquad Y_s^{(l)} = \sum A_s^*(t)\, y_1^{k_1} \ldots y_n^{k_n} \tag{1.6}$$

where some or all $\varkappa_1, \ldots, \varkappa_{n-1}$ may be equal to zero. The asterisk replaces the index $(k_1, \ldots, k_n)$ and the coefficients $A_s^*(t)$ are almost-periodic functions of t, which can be represented by finite Fourier series with arbitrary frequency spectra. For any such function $f(t)$, we have

$$\int f(t)\,dt = gt + \varphi(t), \qquad g = \lim_{t\to\infty} \frac{1}{t}\int_0^t f(t)\,dt$$

where $\phi(t)$ is almost periodic and of the form [3] of $f(t)$.

2. We shall now show that any system of the type (1.6) can be transformed into another system, which will have constant coefficients accompanying nonlinear terms of arbitrary, although finite order N. Thus the problem of stability of almost-periodic motions in the non-essentially singular cases will be reduced to an equivalent problem on stability of a steady-state motion.

We shall assume that the coefficients of the forms $Y_s^{(l)}$ $(l \leqslant k-1)$ in (1.6) are constant while the coefficients of $Y_s^{(k)}$ are almost-periodic functions of time.

Let us introduce a substitution of the type

$$y_s = x_s + \sum u_s^*(t)\, y_1^{k_1} \ldots y_n^{k_n} \qquad (k_1 + \ldots + k_n = k;\ s = 1, \ldots, n) \tag{2.1}$$

where $u_s^*(t)$ are almost-periodic functions whose structure resembles that of $A_s^*(t)$. Expression (2.1) easily yields

$$y_s = x_s + \sum v_s^*(t)\, x_1^{k_1} \ldots x_n^{k_n} \qquad (k_1 + \ldots + k_n \geqslant k) \tag{2.2}$$

When $k_1 + \ldots + k_n = k$, functions $v_s^*(t)$ are equal to $u_s^*(t)$, while when $k_1 + \ldots + k_n > k$, functions $v_s^*(t)$ become polynomials $u_s^*(t)$ with distinct coefficients $(k_1, \ldots, k_n)$.

System (1.6) now becomes

$$x_s^{\cdot} = \varkappa_{s-1} x_{s-1} + \sum_{l\geqslant 2}^{\infty} X_s^{(l)}(x_1, \ldots, x_n;\ t) \qquad (s = 1, \ldots, n;\ \varkappa_0 = 0) \tag{2.3}$$

where the forms $X_s^{(l)}$ with $l \leqslant k-1$ $(s = 1, \ldots, n)$ are equal to $Y_s^{(l)}$ with x_s replacing y_s and $X_s^{(k)}$ have the form

$$X_s^{(k)}(x_1, \ldots, x_n;\ t) = \sum a_s^*(t)\, x_1^{k_1} \ldots x_n^{k_n} \qquad (k_1 + \ldots + k_n = k;\ s = 1, \ldots, n)$$

Coefficients $a_s^*(t)$ are given by

$$a_s^*(t) = -\frac{du_s^*}{dt} - (k_2+1)\varkappa_1 u_s^{(k_1-1,\, k_2+1,\, k_3, \ldots, k_n)} - (k_3+1)\varkappa_2 u_s^{(k_1,\, k_2-1,\, k_3+1,\, k_4, \ldots, k_n)} -$$
$$- \ldots - (k_n+1)\varkappa_{n-1} u_s^{(k_1, \ldots, k_{n-1},\, k_n+1)} + \varkappa_{s-1} \dot{u}_{s-1} + A_s^*(t) \tag{2.4}$$

Let us first determine the coefficient $a_1^{(0,\ldots,0k)}$ using

$$a_1^{(0\ldots 0k)} = -\frac{du_1^{(0\ldots 0k)}}{dt} + A_1^{(0\ldots 0k)}(t)$$

This equation will have an almost-periodic solution for $u_1^{(0,\ldots,0k)}(t)$ of the form given above, if

$$a_1^{(0,\ldots 0k)} = \lim_{t\to\infty} \frac{1}{t}\int_0^t A_1^{(0\ldots 0k)}(t)\,dt \tag{2.5}$$

Thus the coefficient $a_1^{(0,\ldots,0k)}$ will either be a constant, or in a particular case, zero. Having obtained $a_1^{(0,\ldots,0k)}$ from (2.5), we find $u_1^{(0,\ldots,0k)}$ from

$$u_1^{(0\ldots 0k)} = \int_0^t [A_1^{(0\ldots 0k)}(t) - a_1^{(0\ldots 0k)}]\,dt$$

after which we can determine either $u_1^{(0,\ldots,0,1,k-1)}$ or $u_2^{(0,\ldots,0k)}$. The known magnitudes $k\varkappa_{n-1} u_1^{(0,\ldots,0k)}$ or $\varkappa_1 u_1^{(0,\ldots,0k)}$ must be related to the known coefficients $A_1^{(0,\ldots,0,1,k-1)}$ or $A_2^{(0,\ldots,0k)}$. The respective coefficients $a_1^{(0,\ldots,0,1,k-1)}$ or $a_2^{(0,\ldots,0k)}$ will again be constant (or zero).

Continuing this process we shall find, that the k-th order forms appearing in (2.3) will have constant coefficients, while all $u_s^*(k_1+\ldots+k_n=k)$ will be almost-periodic functions of the same structure as $A_s^*(t)$.

Putting k equal to 2, 3,..., N, we shall obtain a system in which all the forms up to and including the N-th order, will have constant coefficients, and this allows us to formulate the following theorem:

T h e o r e m 2.1. If the system of Eqs. (1.1) is such that:

1) Eq. $|p_{ij} - \delta_{ij}\varkappa| = 0$ has p zero roots, q pairs of purely imaginary roots and k roots with negative reals parts;

2) functions X_i are holomorphic in $x_1,\ldots,x_m$ and become zero when $x_1=\ldots=x_m=0$ while the coefficients of their expansions in powers of x_i are almost-periodic functions of t and can be represented by finite Fourier series with arbitrary frequency spectra, then, in the case with nonessential singularity the investigation of the stability of this system always leads to the equivalent problem of stability of a steady-state motion in the critical case of $(p+2q)$ zero roots.

N o t e: Analogous results can be obtained after transforming (1.2) into (1.6), by reducing (1.6) to the, so called, standard form and applying the Krylov-Bogoliubov [4] transformation.

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